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Permissible Bounds on the Coefficients of Approximating Polynomials*

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1. INTRODUCTION

In this paper we consider bounds on the coefficients of algebraic polynomials which approximate continuous functions on a closed interval in the uniform norm. If $f \in C[a, b]$, we write $\|f\| = \max_{a \leq x \leq b} |f(x)|$.

J. D. Stafney [5] proved the following

THEOREM A. *Let $f \in C[0, 1]$, $f(0) = 0$. Let $\eta > 0$ and let $(w_k)_{k=0}^{\infty}$ be any sequence of positive numbers with the property $w_k^{1/k} \to \infty$. Then there exist polynomials $P_n(x) = \sum_{k=0}^{n} a_{nk} x^k$ with $|a_{nk}| < \eta w_k$, $k = 0, 1, 2,\ldots$, such that $\|f - p_n\| < \eta$.*

We note, furthermore, that the η in the inequality $|a_{nk}| < \eta w_k$ adds nothing to Theorem A. Assume, for example, that the theorem was stated with $|a_{nk}| < w_k$. Let $\eta > 0$ and (w_k) be given. Define

$$u_k = \eta w_k, \qquad k = 0, 1, 2, 3,\ldots .$$

Then we have $|a_{nk}| < u_k = \eta w_k$, $k = 0, 1, 2, 3,\ldots$. We may always choose $a_{n0} = 0$. Theorem 3 of this paper shows that $w_k^{1/k} \to \infty$ is not a necessary condition for the conclusion of Theorem A to hold. On the other hand, Stafney [5] shows that $\overline{\lim}_{k\to\infty} w_k^{1/k} < +\infty$ is not sufficient. Hence, it is an interesting problem to ask for necessary and sufficient conditions on the sequence (w_k) for having a theorem like Theorem A.

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The present paper employs Bernstein polynomials to study this problem. This gives a simpler approach and some stronger results than Stafney presents. This method also gives good upper bounds for $\sum_{k=0}^{n} | a_{nk} |$. Also, a strong result is obtained in the special case where $f(x) = 0$ on $[0, c]$ $(0 < c \leqslant 1)$.

2. The Main Theorems

We start with the following theorem, which, together with its proof, will be used below.

Theorem 1. *Suppose $a \leqslant 0 < 1 \leqslant b$ and $f \in C[a, b]$. If $P_n(x) = \sum_{k=0}^{n} a_{nk} x^k$ is the Bernstein polynomial of order n of f, on $[a, b]$, then, for $n = 1, 2, \ldots$,*

$$\| P_n - f \| \leqslant C\omega \left(f, \frac{b-a}{\sqrt{n}}\right), \tag{1}$$

$$\sum_{k=0}^{n} | a_{nk} | \leqslant \| f \| \left(1 + \frac{2}{b-a}\right)^n \tag{2}$$

(ω is the modulus of continuity of f on $[a, b]$.)

Proof. (1) is a well-known result (see [2]).

The Bernstein polynomial of order n of f is given by

$$\text{(i)} \quad B_n(f, x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} f\left(\frac{k}{n}(b-a) + a\right)\binom{n}{k} (x-a)^k (b-x)^{n-k}.$$

By the binomial theorem, we have

$$\text{(ii)} \qquad (x-a)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a^{k-j} x^j,$$

$$(b-x)^{n-k} = \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^{n-k-i} b^i x^{n-k-i}.$$

Substituting (ii) in (i) and rearranging the sum, we have [letting $Z_{nk} = k/n(b-a) + a$]:

$$\text{(iii)} \qquad B_n(f, x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=0}^{n-k} \left[f(Z_{nk}) \binom{n}{k} \cdot \binom{k}{j}\binom{n-k}{i} (-1)^{n-j-i} a^{k-j} b^i \right] x^{n+j-k-i}.$$

If we let

$$P_n(x) = \sum_{k=0}^{n} a_{nk} x^k = B_n(f, x),$$

we see from (iii) that

$$\begin{aligned}
\sum_{k=0}^{n} | a_{nk} | &\leqslant \frac{1}{(b-a)^n} \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=0}^{n-k} | f(Z_{nk})| \binom{n}{k}\binom{k}{j}\binom{n-k}{i} | a |^{k-j} b^i \\
&\leqslant \frac{\|f\|}{(b-a)^n} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k}\binom{k}{j} | a |^{k-j} \sum_{i=0}^{n-k} \binom{n-k}{i} b^i \\
&= \frac{\|f\|}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} | a |^{k-j} (1+b)^{n-k} \\
&= \frac{\|f\|}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} (1 + | a |)^k (1+b)^{n-k} \\
&= \frac{\|f\|}{(b-a)^n} (2 + | a | + b)^n \\
&= \frac{\|f\|}{(b-a)^n} (2 + b - a)^n \\
&= \|f\| \left(1 + \frac{2}{b-a}\right)^n.
\end{aligned}$$

That is,

(iv) $$\sum_{k=0}^{n} | a_{nk} | \leqslant \|f\| \left(1 + \frac{2}{b-a}\right)^n.$$

Theorem 2 now follows from Theorem 1 by considering the rates of convergence of Bernstein polynomials ([2], pp. 20, 21).

THEOREM 2. *Let* $(\eta_n)_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ *be sequences of positive numbers such that* $\delta_n \downarrow 0$ and $n^{1/2}\eta_n \to \infty$. *Assume, furthermore, that* $0 < \eta_n \leqslant 2$ *for all* n. *Let* $f \in C[0, 1]$ *satisfy* $f(0) = 0$. *Then there exist polynomials*

$$P_n(x) = \sum_{j=0}^{n} a_{nj} x^j, \qquad n = 1, 2, 3, \ldots,$$

with the properties:

$$P_n(x) \to f(x), \qquad \text{uniformly on} \quad [0, 1], \tag{3}$$

$$a_{nj} = 0, \qquad \text{for} \quad 0 \leqslant j < n\, \delta_n \eta_n / 2, \tag{4}$$

$$\sum_{j=0}^{n} | a_{nj} | \leqslant \|f\| (1 + \eta_n)^n. \tag{5}$$

Proof. Extend f to a continuous function g on $[0, +\infty)$, by setting

$$g(x) = \begin{cases} f(x), & 0 \leqslant x \leqslant 1, \\ f(1), & 1 \leqslant x. \end{cases}$$

Then f and g have the same norm and the same modulus of continuity ω. For each n, let $b_n = 2/\eta_n$ and consider the polynomials

$$\text{(i)}' \qquad P_n(x) = \frac{1}{b_n{}^n} \sum_{\delta_n \leqslant kb_n/n \leqslant b_n} g\left(\frac{kb_n}{n}\right)\binom{n}{k} x^k(b_n - x)^{n-k}.$$

[If $\delta_n > b_n$, then define $P_n(x) = 0$]. (i)′ may be written in terms of the Bernstein polynomials $B_n(g, x)$ on $[0, b_n]$ as

$$\text{(ii)}' \quad P_n(x) = B_n(g, x) - \frac{1}{b_n{}^n} \sum_{0 \leqslant kb_n/n < \delta_n} g\left(\frac{kb_n}{n}\right)\binom{n}{k} x^k(b_n - x)^{n-k}$$

$$= B_n(g, x) - Q_n(x).$$

But

$$| Q_n(x)| \leqslant \frac{M_n}{b_n{}^n} \sum_{0 \leqslant kb_n/n < \delta_n} \binom{n}{k} x^k(b_n - x)^{n-k}$$

$$\leqslant \frac{M_n}{b_n{}^n} \sum_{k=0}^{n} \binom{n}{k} x^k(b_n - x)^{n-k} = M_n ,$$

where

$$M_n = \max_{0 \leqslant y \leqslant \delta_n} | g(y)| \leqslant \omega(\delta_n).$$

So,

$$| Q_n(x)| \leqslant \omega(\delta_n) \qquad \text{for} \quad 0 \leqslant x \leqslant b_n ,$$

and certainly for $0 \leqslant x \leqslant 1$. Clearly, then,

$$\| Q_n \| \leqslant \omega(\delta_n) \to 0.$$

Now,

$$| g(x) - B_n(g, x)| \leqslant C\omega\left(\frac{b_n}{n^{1/2}}\right), \qquad 0 \leqslant x \leqslant b_n .$$

But

$$\omega(b_n n^{-1/2}) = \omega(2\eta_n^{-1} n^{-1/2}) \to 0.$$

So,

$$\| f - B_n(g, x)\| \leqslant C\omega(2\eta_n^{-1} n^{-1/2}) \to 0.$$

Hence,

$$\|f - P_n\| \leqslant \|f - B_n(g, x)\| + \|Q_n\| \leqslant C\omega(2\eta_n^{-1}n^{-1/2}) + \omega(\delta_n) \to 0.$$

Moreover, if we write $P_n(x) = \sum_{k=0}^{n} a_{nk}x^k$, we see from (i)$'$ that $a_{nk} = 0$ if $k < n\,\delta_n\eta_n/2$. It also follows as in Theorem 1 that

$$\sum_{k=0}^{n} |a_{nk}| \leqslant \|f\| \left(1 + \frac{2}{b_n}\right)^n = \|f\| (1 + \eta_n)^n.$$

This proves the theorem.

THEOREM 3. *Let f be as in Theorem A. The conclusion is still valid if we only assume that there exist, for each $1 > \delta > 0$ and $M > 0$, arbitrarily large $N = N(\delta, M)$ for which $w_k^{1/k} \geqslant M$ if $N\delta \leqslant k \leqslant N$.*

Proof. Let $\eta > 0$ be given $(1 > \eta > 0)$. Choose δ $(1 > \delta > 0)$ such that $0 \leqslant x < \delta$ implies $|f(x)| < \eta/2$. Define

$$Q_n(x) = \sum_{n\delta \leqslant k \leqslant n} f(k/n) \binom{n}{k} x^k(1 - x)^{n-k}$$
$$= \sum_{n\delta \leqslant i \leqslant n} b_{ni}x^i, \qquad n = 1, 2, 3, \ldots .$$

By an argument similar to that used in the proof of Theorem 2, we see that

$$|f(x) - Q_n(x)| < \eta$$

for $0 \leqslant x \leqslant 1$ and for n sufficiently large, and

$$\sum_{i=0}^{n} |b_{ni}| \leqslant \|f\| 3^n, \qquad n = 1, 2, \ldots .$$

It follows from this that

$$|b_{ni}| \leqslant \|f\| 3^{i/\delta} \quad \text{if} \quad \delta n \leqslant i \leqslant n, \quad \text{and if} \quad i < \delta n \quad \text{then} \quad b_{ni} = 0. \quad \text{(i)}''$$

Now let n be one of the numbers $N(\delta, M)$, where $M = 2 \cdot 3^{1/\delta}$. Then (i)$''$ gives

$$|b_{ni}| \leqslant \|f\| \left(\frac{3^{1/\delta}}{w_i^{1/i}}\right)^i w_i \leqslant \|f\| 2^{-i}w_i \leqslant \|f\| 2^{-\delta n}w_i$$

if $\delta n \leqslant i \leqslant n$. In addition to the previous assumptions we take n so large that $\|f\| 2^{-\delta n} < \eta$. This gives $|b_{ni}| \leqslant \eta w_i$, for all i.

THEOREM 4. *Let $f \in C[0, 1]$ and suppose that $f(x) = 0$ on $[0, c]$ $(0 < c \leqslant 1)$. Then there is a sequence of polynomials $\{P_n\}$ $(P_n(x) = \sum_{nc<k\leqslant n} a_{nk}x^k)$ such that*

$$P_n \to f, \qquad \text{uniformly on} \quad [0, 1], \tag{6}$$

and

$$| a_{nk} | \leqslant \|f\| \, 3^{k/c}. \tag{7}$$

Proof. Let

$$P_n(x) = B_n(f, x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k(1-x)^{n-k}$$

be the Bernstein polynomial of order n of f. Since $f(k/n) = 0$ if $(k/n) \leqslant c$, we may write

$$\begin{aligned} P_n(x) &= \sum_{nc<k\leqslant n} f(k/n) \binom{n}{k} x^k(1-x)^{n-k} \\ &= \sum_{nc<k\leqslant n} a_{nk}x^k. \end{aligned}$$

If, in Theorem 1, we let $b = 1$, and $a = 0$, (2) gives

$$\sum_{nc<k\leqslant n} | a_{nk} | \leqslant \|f\| \, 3^n.$$

Hence,

$$| a_{nk} | \leqslant \|f\| \, 3^n \leqslant \|f\| \, 3^{k/c}$$

for each k with $nc < k \leqslant n$.

Remarks

No claim is made that the results contained herein are the best possible. It is, however, remarkable how one can easily apply the Bernstein polynomials to the problem and obtain Theorem A as well as some additional information.

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